Quantum dissipative effects in moving mirrors: a functional approach

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We use a functional approach to study various aspects of the quantum effective dynamics of moving, planar, dispersive mirrors, coupled to scalar or Dirac fields, in different numbers of dimensions. We first compute the Euclidean effective action, and use it to derive the imaginary part of the 'in-out' effective action. We also obtain, for the case of the real scalar field in $1+1$ dimensions, the Schwinger-Keldysh effective action and a semiclassical Langevin equation that describes the motion of the mirror including noise and dissipative effects due to its coupling to the quantum fields.

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I. INTRODUCTION

In the presence of a moving, accelerated mirror, the electromagnetic field evolves from the vacuum to an excited state, containing a non-vanishing number of photons. This 'motion induced radiation' or 'Dynamical Casimir Effect' (DCE) has been the subject of intense theoretical research since its discovery in the seventies [1, 2]. While this phenomenon was initially regarded as being of just theoretical interest (for example as a toy model for black hole evaporation), in recent years it has been pointed out that the experimental verification of the DCE might not be, after all, so far out of reach [3, 4].

Indeed, taking advantage of parametric resonance amplification, this effect could be dramatically increased [5], since the number of photons created within a cavity with a moving mirror should grow exponentially at resonance (i.e., when the mirror's oscillatory frequency doubles one of the eigenfrequencies of the unperturbed cavity). For the case of microwave cavities, the mechanical frequency of the mirror should, however, be extremely high ($\sim$ 1GHz) for this to happen, and this poses the main stumbling block for an experimental verification of the effect.

It has also been suggested that the DCE could be measured in experiments in which the moving mirror is replaced by a semiconductor slab which suddenly changes its conductivity due to illumination with short laser pulses [6]. Unfortunately, the unavoidable losses in the semiconductor could put the viability of this proposal in jeopardy [7]. Yet another alternative that has been advanced [8], which amounts to consider an array of nanoresonators, moving coherently at frequencies in the GHz range. The detection of the created photons could, in this case, be performed using an inverted population of Rydberg atoms.

From the theoretical point of view, the DCE has been analyzed for a variety of geometries and using many different theoretical tools. A particularly interesting functional approach has been proposed by Golestanian and Kardar [9]. They introduce auxiliary fields in the functional integral for the quantum field, whose role is to impose the boundary conditions on the mirrors. This method has been successfully applied, for example, to the calculation of the Euclidean effective action for one and two (slightly deformed) moving mirrors in $d+1$ dimensions [10], deriving also the effective equation of motion for the mirror by analytic continuation of the Euclidean effective action.

In view of the possibility of detecting the DCE using nanoresonators [8], it is of interest to extend this formalism in several directions. On the one hand, it is important to generalize the method, in order to be able to consider dispersive mirrors, rather than just perfectly conducting ones. On the other hand, since the nanoresonators could eventually show quantum behaviour [11], it is worthwhile to consider their quantum to classical transition, and to describe their effective dynamics in terms of a semiclassical Langevin equation.

This paper is a step in that direction [12].

Besides, to exhibit the quite general nature of the phenomenon, it is also interesting to extend the formalism to consider mirrors coupled to different fields, like the case of a moving wall that imposes bag conditions on a Dirac field. In this article, we first show how to generalize the functional approach of [9], to calculate the Euclidean effective action for moving dispersive mirrors coupled to real scalar and then to Dirac fields. We also show how the case of a relativistic mirror also fits in the formalism, by performing minor modifications. We then compute the Schwinger-Keldysh or Closed Time Path (CTP) effective action for a mirror coupled to a scalar field. More realistic situations (a cavity with two no-flat mirrors coupled to the electromagnetic field) will be considered in a forthcoming publication.

This article is organized as follows: in section II, we use a path-integral approach to evaluate the Euclidean effective action for a single, perfect or imperfect, non-relativistic moving mirror in $1+1$ dimensions, both for the real scalar and Dirac field cases. By 'perfect mirror' we mean one that imposes Dirichlet boundary conditions, when coupled to a scalar field, or bag conditions in the Dirac field case. In both cases, the boundary conditions due to the perfect mirror are introduced by the coupling of the quantum field to a singular mass term, localized on a region of codimension 1, with a divergent coupling constant $\lambda \to \infty$. The imperfect mirrors that we shall consider here will be, on the other hand, described by the same kind of interaction term, albeit with a *finite* coupling constant λ.

The changes needed to cope with the relativistic mirror generalization are also presented, taking the real scalar field case as a concrete example, and evaluating the corresponding effective action.

In section III, we consider the case of a (flat) moving mirror in $d+1$ dimensions, coupled to a real scalar field, evaluating explicitly the Euclidean effective action. In section IV, we evaluate and interpret the imaginary part of the in-out effective action, obtained after Wick rotating to real time, for the case of the real scalar field with perfect boundary conditions.

In section V, we evaluate the quantum corrections to the mirrors' real-time equations of motion. In order to do this, we compute the CTP effective action and obtain a semiclassical stochastic equation for the mirror. Moreover, from the imaginary part of the CTP effective action we provide an estimation of the decoherence time for the mirror.

II. MOVING MIRRORS IN $1+1$ DIMENSIONS

A. Real scalar field

We shall begin by considering a massive real scalar field φ coupled to an imperfect mirror, whose position is described by a function $q(x^0)$, so that the real-time Lagrangian density, $\mathcal{L}$ is:

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{2}\,V(x^0,x^1)\,\varphi^2 \;, \tag{1}$$

where V is a δ-like singular function:

$$V(x^0,x^1) \;=\; \lambda\,\delta(x^1 - q(x^0)) \;, \tag{2}$$

determined by the mirror's position. λ is a positive coupling constant. The coupling to the singular field has the effect of introducing a perfect mirror (at $x^1 = q(x^0)$) when $\lambda \to +\infty$, since it then enforces the condition $\varphi = 0$ on the points of the spacetime curve $\mathcal{C}$ defined by the points $(x^0,\, q(x^0))$. On the other hand, the imperfect mirror situation is simulated for $0 < \lambda < \infty$ [14].

Let us now perform a Wick rotation: $x^0 = -i\tau$, and calculate the resulting Euclidean effective action $\Gamma[q(\tau)]$ for the mirror, due to the scalar-field vacuum fluctuations, in the functional-integral representation:

$$e^{-\Gamma[q(\tau)]} \;\equiv\; \mathcal{Z}[q(\tau)] \;=\; \int \mathcal{D}\varphi\; e^{-S[\varphi;q]} \;, \tag{3}$$

where $S[\varphi; q]$ is the Euclidean action:

$$S[\varphi; q] \;=\; S_0[\varphi] \;+\; S_{\mathcal{C}}[\varphi; q] \tag{4}$$

with S_0 denoting the free part

$$S_0[\varphi] \;=\; \frac{1}{2} \int d^2x \big(\partial_\mu \varphi \partial_\mu \varphi + m^2 \varphi^2\big) \tag{5}$$

and $S_{\mathcal{C}}$ the coupling to the mirror,

$$S_{\mathcal{C}}[\varphi; q] \;=\; \frac{\lambda}{2} \int d^2x \; \delta(x_1 - q(x_0)) \big[\varphi(x)\big]^2 \;=\; \frac{\lambda}{2} \int d\tau \big[\varphi(\tau, q(\tau))\big]^2 \;. \tag{6}$$

Euclidean coordinates are denoted by x_μ, where $x_0 \equiv \tau$; the metric tensor is the 2×2 identity matrix.

To proceed, we introduce an auxiliary field $\xi(\tau)$, living in $0+1$ dimensions, whose role is to linearize the term $S_{\mathcal{C}}$, which couples the scalar field to the mirror. The resulting expression for $\mathcal{Z}[q(\tau)]$ is:

$$\mathcal{Z}[q(\tau)] \;=\; \int \mathcal{D}\xi \; e^{-\frac{1}{2\lambda} \int d\tau \xi^2(\tau)} \; \mathcal{Z}_0[J_\xi] \tag{7}$$

where $\mathcal{Z}_0$ is the free generating functional:

$$\mathcal{Z}_0[J] \;=\; e^{-W_0[J]} \;=\; \int \mathcal{D}\varphi \; e^{-S_0(\varphi) + i \int d^2x J(x) \varphi(x)} \;, \tag{8}$$

and J_ξ is a current localized on the defect and proportional to the auxiliary field: $J_\xi(x_0, x_1) \equiv \xi(x_0)\, \delta(x_1 - q(x_0))$. Note that (7) reduces to the approach of Golestanian and Kardar [9] when $\lambda \to \infty$, i.e., when a perfect mirror is considered.

Since the integral over φ is a Gaussian, we can immediately write down the explicit form of W_0,

$$W_0[J] \;=\; \frac{1}{2} \int d^2x \int d^2x' \, J(x) \Delta(x - x') J(x') \;, \tag{9}$$

where Δ is the free Euclidean correlation function:

$$\begin{aligned} \Delta(x - y) \;&=\; \langle \varphi(x) \varphi(y) \rangle \\ &=\; \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot (x-y)} \frac{1}{k^2 + m^2} \;. \end{aligned} \tag{10}$$

Thus we derive for $\mathcal{Z}[q(\tau)]$ a 'dimensionally reduced' path integral expression involving just the auxiliary field ξ,

$$\mathcal{Z}[q(\tau)] \;=\; \int \mathcal{D}\xi \; e^{-\frac{1}{2} \int d\tau \int d\tau' \xi(\tau) \mathcal{K}(\tau, \tau') \xi(\tau')} \;, \tag{11}$$

where we have introduced the kernel $K(\tau, \tau')$

$$\mathcal{K}(\tau, \tau') \;=\; \frac{1}{\lambda} \delta(\tau - \tau') + \Delta\big[\tau - \tau', q(\tau) - q(\tau')\big] \;. \tag{12}$$

The ξ-integral, again a Gaussian, allows as to write down the (formal) result for $\mathcal{Z}[q(\tau)]$ as follows:

$$\mathcal{Z}[q(\tau)] \;=\; \big(\det \mathcal{K}\big)^{-\frac{1}{2}} \tag{13}$$

so that

$$\Gamma[q(\tau)] \;=\; \frac{1}{2}{\rm Tr}\big[\ln\mathcal{K}\big] \;. \tag{14}$$

Let us now approximate (14) for small departures with respect to the static mirror case. To that end, we first expand $\mathcal{K}$:

$$\mathcal{K} \;=\; \mathcal{K}_0 \,+\, \mathcal{K}_1 \,+\, \mathcal{K}_2 \,+\, \ldots \tag{15}$$

where the subscripts denote the order of the corresponding term. To derive the linearized form of the equations of motion, it shall be sufficient to keep terms of up to the quadratic order. It is quite straightforward to see that

$$\mathcal{K}_0(\tau,\tau') \;=\; \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')}\widetilde{\mathcal{K}}_0(\omega) \;,\quad \widetilde{\mathcal{K}}_0(\omega) \;=\; \frac{1}{\lambda} \,+\, \frac{1}{2\sqrt{\omega^2+m^2}} \;, \tag{16}$$

$$\mathcal{K}_2(\tau,\tau') \;=\; \frac{1}{4}\big(q(\tau)-q(\tau')\big)^2 \int \frac{d\omega}{2\pi}\, e^{i\omega(\tau-\tau')}\sqrt{\omega^2+m^2} \;, \tag{17}$$

and that $\mathcal{K}_1$ vanishes. It should be kept in mind that $q(\tau)$ is the *departure* with respect to a constant (fixed to 0 by a shift of the axis, if necessary). This implies, in particular, that its Fourier transform $\tilde{q}(\omega)$ will verify $\tilde{q}(0)=0$. Of course, $\tilde{q}(0)=0$ alone does not imply a small departure. Indeed, the condition holds true for some motions that correspond to an unbounded motion, like $\tilde{q}(\omega)\propto i\delta^{'''}(\omega)$, which comes from $q(\tau)\propto\tau^3$. But in this case the quadratic approximation fails, since $q(\tau)$ becomes large (and $\tilde{q}(\omega)$ singular).

Coming back to the expression for $\Gamma[q(\tau)]$, expanding up to the second order in the fluctuation, and discarding a $q(\tau)$-independent term, we see that

$$\Gamma[q(\tau)] \;=\; \frac{1}{2}\,{\rm Tr}\ln\big[\mathcal{K}_0+\mathcal{K}_2\big] \;\simeq\; \frac{1}{2}{\rm Tr}\big[\mathcal{K}_0^{-1}\mathcal{K}_2\big] \;\equiv\; \Gamma_2[q(\tau)] \;, \tag{18}$$

where $\Gamma_2[q(\tau)]$ may be written more explicitly as follows:

$$\Gamma_2[q(\tau)] \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \big[\mathcal{K}_0^{-1}(\tau,\tau')\mathcal{K}_2(\tau',\tau)\big] \;. \tag{19}$$

Using the explicit form for $\mathcal{K}_0$ and $\mathcal{K}_2$,

$$\Gamma_2[q(\tau)] \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' (q(\tau)-q(\tau'))^2 F(\tau-\tau') \;, \tag{20}$$

where

$$F(\tau-\tau') \;=\; \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')}\tilde{F}(\omega)\,, \tag{21}$$

with

$$\tilde{F}(\omega) \;=\; \frac{1}{4}\int \frac{d\nu}{2\pi}\,\big[\frac{1}{\lambda} \,+\, \frac{1}{2\sqrt{(\nu+\omega)^2+m^2}}\big]^{-1}\sqrt{\nu^2+m^2} \;. \tag{22}$$

It is clear that we may subtract from $\tilde{F}(\omega)$ its value at zero-frequency, since any ω-independent part would give zero when inserted in $\Gamma_2[q(\tau)]$ (it would produce a $\delta(\tau-\tau')$ contribution to F, multiplied by a continuous function that vanishes when $\tau=\tau'$). Thus we introduce

$$\tilde{F}_s(\omega) \;\equiv\; \tilde{F}(\omega) \,-\, \tilde{F}(0)\,, \tag{23}$$

the subtracted version of $\tilde{F}$. Since $\tilde{F}_s(0) = 0$, we obviously have $\int d\tau F_s(\tau) = 0$, and

$$\begin{aligned}
\Gamma_2[q(\tau)] &= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' (q(\tau) - q(\tau'))^2 F(\tau - \tau') \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' (q(\tau) - q(\tau'))^2 F_s(\tau - \tau') \\
&= -\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' q(\tau) q(\tau')\, F_s(\tau - \tau') \; . \qquad (24)
\end{aligned}$$

Expression (22) for $\tilde{F}$ is divergent; to regulate it we introduce a symmetric frequency cutoff Ξ, such that $|\nu| \leq \Xi$, and the regulated function $\tilde{F}_s(\omega, \Xi)$ is:

$$\begin{aligned}
\tilde{F}_s(\omega, \Xi) &= \frac{1}{4}\int_{-\Xi}^{\Xi} \frac{d\nu}{2\pi} \Big\{ \big[\frac{1}{\lambda} + \frac{1}{2\sqrt{(\nu+\omega)^2 + m^2}}\big]^{-1} \sqrt{\nu^2 + m^2} \\
&- \big[\frac{1}{\lambda} + \frac{1}{2\sqrt{\nu^2 + m^2}}\big]^{-1} \sqrt{\nu^2 + m^2} \Big\} \; . \qquad (25)
\end{aligned}$$

$\tilde{F}_s(\omega, \Xi)$ in (25) is convergent for $\Xi \to \infty$, so this 'symmetric-limit' regularization yields a finite value for $\tilde{F}_s(\omega) \equiv \lim_{\Xi\to\infty} \tilde{F}_s(\omega, \Xi)$ when the regulator is removed. Unfortunately, there seems to be no analytic expression for $\tilde{F}_s(\omega)$ which is valid for arbitrary values of the constants m and λ. We can, however, calculate it for different relevant particular cases:

1. $m = 0$, $\lambda \to \infty$: in this case, we have

$$\begin{aligned}
\tilde{F}_s(\omega) &= \lim_{\Xi\to\infty} \frac{1}{2}\int_{-\Xi}^{\Xi} \frac{d\nu}{2\pi} \Big[|\nu + \omega||\nu| - |\nu|^2\Big] \\
&= \frac{1}{12\pi}|\omega|^3 \; . \qquad (26)
\end{aligned}$$

A cubic dependence in ω could have been guessed on dimensional grounds. The numerical coefficient coincides with previously obtained results [13].

2. $m = 0$, $\lambda < \infty$: a property that we immediately see is that, for any finite λ, the large-ν behaviour of the integral is improved (by a power of ν) with respect to the perfect mirror ($\lambda \to \infty$) case. As a consequence, the result obtained by taking the $\Xi \to \infty$ and $\lambda \to \infty$ limits will depend on the order in which they are taken.

The ν integral (for a finite λ) and its $\Xi \to \infty$ limit can be evaluated exactly in this case, the result being,

$$\tilde{F}_s(\omega) = \frac{\lambda^2}{16\pi^2}\Big[2|\omega| - \lambda\big(1 + \frac{2}{\lambda}|\omega|\big)\ln\big(1 + \frac{2}{\lambda}|\omega|\big)\Big] \; . \qquad (27)$$

Performing a large-λ expansion in the previous expression; we see that

$$\tilde{F}_s(\omega) = -\frac{\lambda}{8\pi}\omega^2 + \frac{1}{12\pi}|\omega|^3 + \mathcal{O}(\lambda^{-1}) \; . \qquad (28)$$

The second term is independent of λ, and it coincides with the result Eq.(26) obtained for $\lambda \to \infty$. The first term was absent from the perfect mirror case, and is a reflection of the fact that, as anticipated, the $\Xi \to \infty$ and $\lambda \to \infty$ limits do not commute. The reason for that difference is that the finite-λ system includes the effect of more quantum fluctuations than in the infinite-λ case. The resulting difference

between the results obtained for those different limits has, however, a simple physical interpretation. Indeed, that difference $\delta\tilde{F}_s$ comes from the $\mathcal{O}(\lambda)$ term in Eq.(28):

$$\delta\tilde{F}_s(\omega) \;\equiv\; -\frac{\lambda}{8\pi}\omega^2\,, \tag{29}$$

a term which, when inserted into the expression for $\Gamma_2[q]$ yields

$$\delta\Gamma_2[q] \;=\; \int d\tau\frac{1}{2}\,\mu(\lambda)\dot{q}^2(\tau)\;, \tag{30}$$

where $\mu(\lambda) = \frac{\lambda}{4\pi}$. This term can, of course, be regarded as a renormalization in the mirror's mass, when the mirror has a non-relativistic kinetic-energy term, as it is usually assumed. It shouldn't come as a surprise that the outcome of the calculation is a non-relativistic invariant object: the coupling between mirror and field, $S_{\mathcal{C}}$, does in fact assume a non-relativistic description for the mirror, since it is not a relativistic invariant. The covariant formulation of this example is presented, for the sake of completeness, in II B.

3. $m \neq 0$, $\lambda \to \infty$: the exact result for this case can also be obtained, although the calculation is more involved. As outlined in Appendix A, the final result is:

$$\tilde{F}_s(\omega) = \frac{1}{12\pi}\int_0^1 \frac{d\alpha}{\alpha(\alpha-1)}\left[\left[\alpha(\alpha-1)\omega^2 + m^2\right]^{3/2} - m^3\right] \tag{31}$$

which reduces to the proper result Eq.(26) in the $m \to 0$ limit.

B. Real scalar field: relativistic mirror

We present here a relativistically invariant formulation of the real scalar field case. The main reason, besides its intrinsic interest, is that it makes it easier to understand the approximation incurred in the non relativistic approach we (implicitly) used in the previous subsections. This problem has been considered previously in Ref. [15] using a canonical formalism.

An explicitly invariant coupling can be constructed, for example, by considering a relativistic generalization of coupling term, $S_{\mathcal{C}} \to S_{\mathcal{C}}^{\rm rel}$:

$$S_{\mathcal{C}}^{\rm rel}[\varphi,q] \;=\; \frac{\lambda}{2}\int d^2x \int ds\sqrt{\dot{q}_\mu(s)\dot{q}_\mu(s)}\,\delta^{(2)}[x-q(s)]\big[\varphi(x)\big]^2 \tag{32}$$

where $q_\mu(s)$, $\mu = 0,1$, is a suitable parametrization of the worldline described by the mirror. We use the notation $\dot{q}_\mu = \frac{dq_\mu}{ds}$.

When the parametrization is such that s coincides with the laboratory time, $x_0 \to (x_0, q(x_0))$, we obtain:

$$S_{\mathcal{C}}^{\rm rel}[\varphi,q] \;=\; \frac{\lambda}{2}\int dx_0\sqrt{1+\dot{q}^2(x_0)}\,\big[\varphi(x_0,q(x_0))\big]^2\,, \tag{33}$$

which indeed reduces to the non-relativistic term $S_{\mathcal{C}}$ for $|\dot{q}| << 1$, and justifies *a posteriori* the non-relativistic coupling when that condition is fulfilled.

Let us now write down the expressions for the (Euclidean) relativistic versions of the objects we have considered before:

$$\mathcal{Z}^{\rm rel}[q(s)] \;=\; \int \mathcal{D}\xi\; e^{-\frac{1}{2\lambda}\int ds\xi^2(s)}\;\mathcal{Z}_0[J_\xi^{\rm rel}] \tag{34}$$

where

$$J_\xi^{\rm rel}(x_0,x_1) \equiv \int ds \xi(s)\, |\dot{q}(s)|^{\frac{1}{2}} \delta^{(2)}(x - q(s)) \;. \tag{35}$$

Then

$$e^{-\Gamma^{\rm rel}[q(s)]} = \mathcal{Z}^{\rm rel}[q(s)] \;=\; \big(\det \mathcal{K}^{\rm rel}\big)^{-\frac{1}{2}} \tag{36}$$

where

$$\mathcal{K}^{\rm rel}(s,s') \;=\; \frac{1}{\lambda}\,\delta(s-s') \,+\, |\dot{q}(s)|^{\frac{1}{2}}\,\Delta\big[q(s)-q(s')\big]\,|\dot{q}(s')|^{\frac{1}{2}} \;. \tag{37}$$

The next step is to perform an expansion in powers of $f_\mu(s)$, the fluctuating part of $q_\mu(s)$:

$$q_\mu(s) \;=\; q_\mu^{(0)}(s) \,+\, f_\mu(s) \tag{38}$$

where $q_\mu(s)$ is analogous to the worldline for the 'static' mirror. It is in fact a linear function of s

$$q_\mu^{(0)}(s) \;=\; a_\mu \,+\, v_\mu\, s \tag{39}$$

where a_μ and v_μ are constant vectors; v_μ is time-like in the real time description. The fluctuating part, f_μ, will be assumed to be such that $\dot{f}_\mu$ is orthogonal to v_μ. The reason is that parallel components amount to fluctuations in the parametrization, which are of course irrelevant in a reparametrization-invariant theory.

In order to simplify matters, we use in what follows a specific convariant parametrization; namely, we assume that s is the mirror's proper time. Then the $|\dot{q}|^{\frac{1}{2}}$ factors become both equal to 1. The calculations then proceed, for this parametrization, in a way that mimics the non-relativistic ones. The quadratic part of the effective action is given by,

$$\Gamma_2^{\rm rel}[q(s)] \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds' \big[(\mathcal{K}^{\rm rel})_0^{-1}(s,s')\mathcal{K}_2^{\rm rel}(s',s)\big] \;. \tag{40}$$

where

$$\begin{aligned}
\mathcal{K}_0^{\rm rel}(s,s') \;&=\; \frac{1}{\lambda}\,\delta(s-s') \,+\, \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot v(s-s')}\,\frac{1}{k^2+m^2} \\
&=\; \int \frac{d\omega}{2\pi}\, e^{i\omega(s-s')}\frac{1}{\lambda} \,+\, \int \frac{d\omega}{2\pi}\, e^{i\omega|v|(s-s')}\frac{1}{2\sqrt{\omega^2+m^2}} \\
&=\; \int \frac{d\omega}{2\pi}\, e^{i\omega(s-s')}\Big[\frac{1}{\lambda} \,+\, \frac{1}{2|v|\,\sqrt{\omega^2+m^2}}\Big]
\end{aligned} \tag{41}$$

and

$$\mathcal{K}_2^{\rm rel}(s,s') \;=\; \frac{1}{4}\big[f(s)-f(s')\big]^2\,\frac{1}{|v|}\int \frac{d\omega}{2\pi}\, e^{i\omega(s-s')}\,\sqrt{\omega^2+m^2} \;. \tag{42}$$

Then

$$\Gamma_2^{\rm rel}[q(s)] \;=\; -\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds' q(s)\, F_s^{\rm rel}(s-s')\, q(s') \;. \tag{43}$$

where:

$$\begin{aligned}
\tilde{F}_s^{\rm rel}(\omega) \;&=\; \lim_{\Xi\to\infty}\frac{1}{4}\int_{-\Xi}^{\Xi}\frac{d\nu}{2\pi}\Big\{\big[\frac{|v|}{\lambda} \,+\, \frac{1}{2\sqrt{(\nu+\omega)^2+m^2}}\big]^{-1}\sqrt{\nu^2+m^2} \\
&\quad -\, \big[\frac{|v|}{\lambda} \,+\, \frac{1}{2\sqrt{\nu^2+m^2}}\big]^{-1}\sqrt{\nu^2+m^2}\Big\} \;.
\end{aligned} \tag{44}$$

This implies, in particular, that in the $\lambda \to \infty$ limit, the results for $\Gamma^{\rm rel}[q]$ coincide with the ones for the non-relativistic case, if one uses the proper-time as the evolution parameter.

It is interesting to consider now the situation when λ is large but finite and $m = 0$. It is quite straightforward to see that this contribution generates, as in the non-relativistic case, an order-λ term:

$$\delta\Gamma_2^{\rm rel}[q] \;=\; \frac{1}{|v|} \int ds \frac{1}{2}\,\mu(\lambda) \dot{f}^2(s) \;, \tag{45}$$

where $\mu(\lambda) = \frac{\lambda}{4\pi}$. On the other hand, since $\dot{q}^2(s) = v^2 + \dot{f}^2$ and we have an equivalent expression for $\delta\Gamma_2^{\rm rel}$:

$$\delta\Gamma_2^{\rm rel}[q] \;=\; \mu^{\rm rel}(\lambda, v) \int ds \;, \tag{46}$$

a term that has a form of a (v-dependent) mass counterterm for the relativistic mirror action, with

$$\mu^{\rm rel}(\lambda, v) \;\equiv\; \frac{1}{2}\mu(\lambda)(\frac{1}{|v|} - |v|) \;, \tag{47}$$

since the natural relativistic action for a mirror with mass M is:

$$S^{\rm rel}_{\rm mirror} \;=\; M \int ds \;. \tag{48}$$

C. Dirac field

Let us now consider the case of a Dirac field with bag-like boundary conditions on the 'mirror'. This kind of boundary condition can also be introduced by means of an interaction with a singular potential; indeed, as shown in [16], bag-like boundary conditions may be introduced by considering the limit of a singular mass term. The real-time Lagrangian density is then

$$\mathcal{L} \;=\; \bar{\psi}\big[i \not\partial - m - V(x^0, x^1)\big]\psi(x) \tag{49}$$

where V is the singular potential defined in (2).

As in the real scalar field case, we may pass to the Euclidean formulation, to calculate $\Gamma[q(\tau)]$ for the mirror:

$$e^{-\Gamma[q(\tau)]} \;=\; \mathcal{Z}[q(\tau)] \;=\; \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S[\bar{\psi},\psi;q]} \;, \tag{50}$$

where now:

$$S[\bar{\psi}, \psi; q] \;=\; S_0[\bar{\psi}, \psi] \,+\, S_{\mathcal{C}}[\bar{\psi}, \psi; q] \tag{51}$$

with

$$S_0[\bar{\psi}, \psi] \;=\; \int d^2x \; \bar{\psi} \; (\not\partial + m)\psi \tag{52}$$

and

$$S_{\mathcal{C}}[\bar{\psi}, \psi; q] \;=\; \lambda \int d^2x \; \bar{\psi}(x) \; \delta[x_1 - q(x_0)] \; \psi(x) \;. \tag{53}$$

Now to linearize the coupling we need two 2-component (Grassmann) auxiliary fields, $\xi(\tau)$ and $\bar{\xi}(\tau)$, so that

$$\mathcal{Z}[q(\tau)] \;=\; \int \mathcal{D}\xi \mathcal{D}\bar{\xi}\, e^{-\frac{1}{\lambda}\int d\tau \bar{\xi}(\tau)\xi(\tau)} \; \mathcal{Z}_0[\bar{\eta}_\xi, \eta_\xi] \;, \tag{54}$$

with

$$\mathcal{Z}_0[\bar{\eta},\eta] \;=\; e^{-W_0[\bar{\eta},\eta]} \tag{55}$$

where

$$W_0[\bar{\eta},\eta] \;=\; \int d^2x \int d^2x' \;\bar{\eta}(x)\;\mathcal{S}_f(x-x')\;\eta(x') \tag{56}$$

and $\mathcal{S}_f$ is the free Dirac propagator:

$$\mathcal{S}_f(x,x') \;=\; \langle \psi(x)\bar{\psi}(x')\rangle \;=\; \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot(x-x')} \frac{1}{i\not{p}+m} \;. \tag{57}$$

We have introduced the sources:

$$\eta_\xi(x) \;=\; \xi(x_0)\,\delta[x_1-q(x_0)] \;\;,\quad \bar{\eta}_{\bar{\xi}}(x) \;=\; \bar{\xi}(x_0)\,\delta[x_1-q(x_0)]\,. \tag{58}$$

Performing the (Grassmann) Gaussian integral over the auxiliary fields, we see that

$$\Gamma[q] \;=\; -\mathrm{Tr}\ln\big[\mathcal{K}_f\big] \tag{59}$$

where:

$$\mathcal{K}_f(\tau,\tau') \;=\; \frac{1}{\lambda}\delta(\tau-\tau') \,+\, \mathcal{S}_f(\tau-\tau', q(\tau)-q(\tau'))\;. \tag{60}$$

We again expand in powers of the fluctuating $q(\tau)$,

$$\mathcal{K}_f \;=\; \mathcal{K}_f^{(0)} \,+\, \mathcal{K}_f^{(1)} \,+\, \mathcal{K}_f^{(2)} \,+\,\ldots \tag{61}$$

with

$$\mathcal{K}_f^{(0)}(\tau-\tau') \;=\; \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')} \widetilde{\mathcal{K}}_f^{(0)}(\omega)\;,\quad \widetilde{\mathcal{K}}_f^{(0)}(\omega) \;=\; \Big[\frac{1}{\lambda} + \frac{-i\gamma_0\omega+m}{2\sqrt{\omega^2+m^2}}\Big]\;, \tag{62}$$

$$\mathcal{K}_f^{(1)}(\tau-\tau') \;=\; -\gamma_1\,[q(\tau)-q(\tau')] \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')} \frac{1}{2}\sqrt{\omega^2+m^2} \tag{63}$$

and

$$\mathcal{K}_f^{(2)}(\tau-\tau') \;=\; \frac{1}{4}\,[q(\tau)-q(\tau')]^2 \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')} \sqrt{\omega^2+m^2}\,(-i\gamma_0\omega+m)\;. \tag{64}$$

Up to second order in the fluctuation,

$$\begin{aligned}
\Gamma[q(\tau)] \;&\simeq\; \Gamma_2[q(\tau)] \\
\Gamma_2[q] \;&=\; -\mathrm{Tr}\Big[(\mathcal{K}_f^{(0)})^{-1}\mathcal{K}_f^{(2)}\Big] + \frac{1}{2}\mathrm{Tr}\Big\{\big[(\mathcal{K}_f^{(0)})^{-1}\mathcal{K}_f^{(1)}\big]^2\Big\} \\
&\equiv\; \Gamma_2^{(1)}[q(\tau)] \,+\, \Gamma_2^{(2)}[q(\tau)]
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
\Gamma_2^{(1)}[q(\tau)] \;&=\; -\mathrm{Tr}\Big[(\mathcal{K}_f^{(0)})^{-1}\mathcal{K}_f^{(2)}\Big] \\
&=\; -\int d\tau \int d\tau' [q(\tau)-q(\tau')]^2\, F^{(1)}(\tau-\tau')
\end{aligned} \tag{66}$$

with

$$\tilde{F}^{(1)}(\omega) = \int \frac{d\nu}{2\pi} \frac{1}{(\lambda^2+4)m^2 + 4\lambda\sqrt{m^2+(\omega+\nu)^2}m + (\lambda^2+4)(\omega+\nu)^2}$$
$$\times \Big[\lambda\sqrt{m^2+\nu^2}(2m^3 + \lambda\sqrt{m^2+(\omega+\nu)^2}m^2$$
$$+ \; 2(\omega+\nu)^2 m + \lambda\nu(\omega+\nu)\sqrt{m^2+(\omega+\nu)^2})\Big] \tag{67}$$

which for the $\lambda \to \infty$ and $m \to 0$ case reduces to:

$$\tilde{F}^{(1)}(\omega) = \int \frac{d\nu}{2\pi} \frac{(\omega+\nu)}{|\omega+\nu|}\nu|\nu| \; . \tag{68}$$

For the remaining term, $\tilde{F}^{(2)}$, a somewhat lengthy calculation shows that it vanishes (for any value of λ and m).

For the special case of $m = 0$ and and $\lambda \to \infty$, the subtracted version of $\tilde{F}^{(1)}(\omega)$ is

$$\tilde{F}^{(1)}_s(\omega) = \frac{2}{3}|\omega|^3 \; . \tag{69}$$

III. PLANE MIRROR COUPLED TO A REAL SCALAR FIELD IN $d+1$ DIMENSIONS

We shall consider here the generalization of the calculations of the previous section, in particular the ones for the real scalar field, to the case of a flat moving mirror in $d+1$ dimensions. The mirror's Euclidean world-volume is defined by the equation

$$x_d - q(x_0) = 0 \; ; \tag{70}$$

the coordinates $x_1, x_2, \ldots, x_{d-1}$ shall be denoted collectively by $x_\parallel$, since they are parallel to the mirror.

In a quite straightforward generalization of the derivation implemented for the $1+1$ dimensional case, we introduce auxiliary fields $\xi(\tau, x_\parallel)$, living in $d-1$ dimensions, obtaining for the $d+1$ dimensional vacuum amplitude, $\mathcal{Z}^{(d+1)}[q(\tau)]$, the expression:

$$\mathcal{Z}^{(d+1)}[q(\tau)] = \int \mathcal{D}\xi \; e^{-\frac{1}{2}\int d\tau \int d\tau' \xi(\tau,x_\parallel)\mathcal{K}(\tau,x_\parallel;\tau',x'_\parallel)\xi(\tau',x'_\parallel)} \; , \tag{71}$$

where

$$\mathcal{K}(\tau, x_\parallel; \tau', x'_\parallel) = \frac{1}{\lambda}\delta(\tau-\tau')\delta(x_\parallel - x'_\parallel) + \Delta\big[\tau-\tau', x_\parallel - x'_\parallel, q(\tau) - q(\tau')\big] \; . \tag{72}$$

The ξ-integral is again Gaussian, and we take advantage of the translation invariance along $x_\parallel$ to Fourier transform with respect to those coordinates, obtaining

$$\Gamma^{(d+1)}[q(\tau)] = L^{d-1} \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \Gamma^{(1+1)}[q(\tau), m(p_\parallel)] \tag{73}$$

where in the last expression we introduced L^{d-1}, the 'area' of the plate, and $\Gamma^{(1+1)}[q(\tau), m(k_\parallel)]$ denotes the effective action for the $1+1$ dimensional case, calculated with a mass depending on the parallel momentum, through the equation:

$$m^2(k_\parallel) = m^2 + k_\parallel^2 \; , \tag{74}$$

with m denoting the standard mass of the field.

The L^{d+1} factor is divergent for an infinite plate. This divergence is, however, harmless from the physical point of view, since the natural object to calculate is not the force but rather the *pressure* experienced by the mirror, hence the area factor is divided out.

In the quadratic approximation we have

$$\frac{1}{L^{d-1}}\,\Gamma_2^{(d+1)}[q(\tau)] \;=\; -\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' q(\tau)q(\tau')\, F_s^{(d+1)}(\tau-\tau')\;, \tag{75}$$

where

$$\begin{aligned}\tilde{F}_s^{(d+1)}(\omega,\Xi) \;=\; & \frac{1}{4}\int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \int_{-\Xi}^{\Xi}\frac{d\nu}{2\pi}\Big\{\big[\frac{1}{\lambda} + \frac{1}{2\sqrt{(\nu+\omega)^2+m^2+k_\parallel^2}}\big]^{-1} \\ & - \big[\frac{1}{\lambda} + \frac{1}{2\sqrt{\nu^2+m^2+k_\parallel^2}}\big]^{-1}\Big\}\sqrt{\nu^2+m^2+k_\parallel^2}\;.\end{aligned} \tag{76}$$

As an example, we consider the particular case $m = 0$ and $\lambda \to \infty$:

$$\begin{aligned}\tilde{F}_s^{(d+1)}(\omega,\Xi) \;=\; & \frac{1}{2}\int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \int_{-\Xi}^{\Xi}\frac{d\nu}{2\pi}\Big\{\sqrt{(\nu+\omega)^2+k_\parallel^2} \\ & - \sqrt{\nu^2+k_\parallel^2}\Big\}\sqrt{\nu^2+k_\parallel^2}\;.\end{aligned} \tag{77}$$

As already mentioned, the calculation in $d+1$ dimensions is similar to the massive case in $1+1$ dimensions. Following the steps described in Appendix A we find

$$\tilde{F}_s^{(d+1)}(\omega) = \frac{\Gamma^2(\frac{1+d}{2})\Gamma(-1-(d/2))}{2^{d+3}\pi^{d/2+1}\Gamma(d+1)}(\omega^2)^{1+\frac{d}{2}} \tag{78}$$

While in an odd number of space dimensions the result could be predicted by dimensional analysis, there is a subtle point in even dimensions. As $\Gamma(-1-(d/2))$ is divergent in this case, it is necessary to introduce in the Lagrangian a counterterm with higher derivatives of the mirror's position. Once the divergence is absorbed, a finite term remains, proportional to $\log[\omega/\mu]$, where μ is an arbitrary constant, determined by the renormalization point.

IV. IMAGINARY PART OF THE IN-OUT EFFECTIVE ACTION

One of the most distinctive signals of the dispersive effects due to an accelerated mirror is a non-vanishing probability of producing a particle pair out of the vacuum. Indeed, the total probability of producing a particle pair when the whole history of the mirror, from $t \to -\infty$ to $t \to +\infty$, is taken into account, can be obtained from the imaginary part of the 'in-out' real time effective action Γ^{io}:

$$P \;=\; 2\,\mathrm{Im}[\Gamma^{io}]\;. \tag{79}$$

This real-time effective action can, on the other hand, be obtained by performing the inverse Wick rotation on the Euclidean $\Gamma[q(\tau)]$ effective actions that we have just calculated, back to real time.

Let us obtain the explicit form of P for two illustrative examples, the cases of the massless and massive real scalar fields in $1+1$ dimensions, since they encode the main features of the physical process we want to describe (other cases will indeed give different results, but they will be kinematical in nature).

The quadratic approximation to the imaginary-time effective action, $\Gamma_2[q(\tau)]$, whose general form in Fourier space is:

$$\Gamma_2[q] \;=\; -\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{F}_s(\omega)\, |\tilde{q}(\omega)|^2 \;, \tag{80}$$

leads to

$$\Gamma_2^{io}[q] \;=\; \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{F}_s(i\omega)\, |\tilde{q}(\omega)|^2 \;, \tag{81}$$

where we kept the same notation for the rotated function q. Thus,

$$P \;=\; 2\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \mathrm{Im}[\tilde{F}_s(i\omega)]\, |\tilde{q}(\omega)|^2 \;. \tag{82}$$

Let us now evaluate P for the two cases mentioned above: in the simplest case of a massless real scalar field with perfect boundary conditions in $1+1$ dimensions, we have,

$$\mathrm{Im}\tilde{F}_s(i\omega) = \frac{1}{12\pi}(-\omega^2)^{3/2} = \pm\frac{1}{12\pi}|\omega|^3 \;. \tag{83}$$

The two signs correspond to the two possible determinations of the square root. Of course, only the positive one corresponds to the right physical situation (Feynman conditions):

$$P \;=\; \frac{1}{6\pi}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, |\omega|^3\, |\tilde{q}(\omega)|^2 \;. \tag{84}$$

In the case $m \neq 0$ and $\lambda \to \infty$, the imaginary part of $\tilde{F}_s$ can also be computed. Indeed, from (31), we see that

$$\mathrm{Im}\tilde{F}_s(i\omega) = \frac{\theta(\omega^2-4m^2)}{12\pi}|\omega^3|\int_0^{1-\frac{4m^2}{\omega^2}} dx\,\big[1-\frac{4m^2}{\omega^2(1-x^2)}\big]^{3/2}. \tag{85}$$

The integral can be computed explicitly in terms of elliptic functions, but we will not need that rather cumbersome expression in what follows, since we want to pinpoint a rather interesting physical phenomenon: the presence of a threshold in the imaginary part of the effective action. This can be understood as follows: if the mirror oscillates with a frequency ω, the reflection of a single field mode, with frequency ω_k, will generate frequency sidebands $\omega_k-\omega$, $\omega_k+\omega$. In order to create particles it is necessary to have a mix of positive and negative frequencies, and the negative frequency should be smaller than $-m$, i.e. $\omega_k-\omega<-m$. This yields $\omega>2m$, as predicted by the previous equation.

V. REAL TIME DYNAMICS: THE SCHWINGER-KELDYSH EFFECTIVE ACTION

Up to now, we have considered the Euclidean effective action that describes the dynamics of the mirror after integration of the quantum fields, and applied it to study the probability of emitting a particle pair during the whole evolution of the system, by performing a Wick rotation back to Minkowski space. The last object is the in-out effective action, which cannot be applied in a straightforward way to the derivation of the equations of motion, since they would become neither real nor causal.

As is well known, in order to get the correct effective equations of motion, one should compute the in-in, Schwinger-Keldysh or Closed Time Path Effective Action (CTPEA) [17], which also has information on the stochastic dynamics of the mirror [18]. The CTPEA is defined as

$$e^{-i\Gamma_{\rm CTP}[q^+,q^-]} = \int \mathcal{D}\phi^+\mathcal{D}\phi^-\, e^{i(S[q^+,\phi^+]-S[q^-,\phi^-])}, \tag{86}$$

and the field equations are obtained taking the variation of this action with respect to the q^+, and then setting $q^+ = q^-$. As in the Euclidean case Eq.(7), one can introduce two auxiliary fields $\xi_\pm(t)$ living in $0+1$ dimensions, in order to linearize the coupling between the mirror and the field. Instead of doing this, we will follow an alternative procedure. Using a more concise notation, we can write the CTPEA as

$$e^{-i\Gamma_{\rm CTP}[q]} = \int \mathcal{D}\phi e^{iS^{\mathcal{C}}[q,\phi]}, \tag{87}$$

where we have introduced the CTP complex temporal path $\mathcal{C}$, going from minus to plus infinity $\mathcal{C}_+$ and backwards $\mathcal{C}_-$, with a decreasing (infinitesimal) imaginary part. Time integration over the contour $\mathcal{C}$ is defined by $\int_{\mathcal{C}} dt = \int_{\mathcal{C}_+} dt - \int_{\mathcal{C}_-} dt$. The field ϕ appearing in Eq.(87) is related to those in Eq.(86) by $\phi(t,\vec{x}) = \phi_\pm(t,\vec{x})$ if $t \in \mathcal{C}_\pm$. The same applies to the mirror's position q.

The equation above is useful because it has the structure of the usual in-out or the Euclidean effective action. Feynman rules are therefore the ordinary ones, replacing Euclidean propagator by [19]

$$G(x,y) = \begin{cases} G_F(x,y) = i\langle 0,in|T\phi(x)\phi(y)|0,in\rangle, & t,t' \text{ both on } \mathcal{C}_+ \\ G_D(x,y) = -i\langle 0,in|\tilde{T}\phi(x)\phi(y)|0,in\rangle, & t,t' \text{ both on } \mathcal{C}_- \\ G_+(x,y) = -i\langle 0,in|\phi(x)\phi(y)|0,in\rangle, & t \text{ on } \mathcal{C}_-, t' \text{ on } \mathcal{C}_+ \\ G_-(x,y) = i\langle 0,in|\phi(y)\phi(x)|0,in\rangle, & t \text{ on } \mathcal{C}_+, t' \text{ on } \mathcal{C}_- \end{cases} \tag{88}$$

Explicitly

$$G_F(x,y) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{ip(x-y)}}{p^2+m^2-i\epsilon} = G_D^*(x,y), \tag{89}$$

$$G_\pm(x,y) = \mp \int \frac{d^{d+1}p}{(2\pi)^{d+1}} e^{ip(x-y)} 2\pi i \delta(p^2-m^2)\theta(\pm p^0). \tag{90}$$

The considerations above will allow us to compute the CTPEA using the Euclidean results of the previous sections. To do this, we will rewrite the Euclidean effective action given in Eq.(24) using a spectral decomposition for the form factor $F_s(\tau-\tau')$. For definiteness we will consider the concrete example of a dispersive mirror (finite λ) coupled to a massless scalar field.

The Euclidean effective action can be rewritten as

$$\Gamma_2[q(\tau)] = -\frac{\lambda^2}{4\pi^3}\int d\tau \int d\tau' q(\tau)q(\tau') \int_0^\infty dz\,(1-f(2z/\lambda))\frac{d^2}{d\tau^2}G_{\rm E}(\tau-\tau',z^2),$$

where $G_E(\tau,z^2)$ is 0+1 Euclidean propagator with mass z^2 and

$$f(z) = \frac{\arctan z}{z} + \frac{1}{2}\ln(1+z^2) \tag{91}$$

(see Appendix B for details).

The CTPEA can be obtained from the Euclidean one considering the contour $\mathcal{C}$ and replacing $G_E(z,t)$ according to the rules given in Eq.(88). The result is

$$\begin{aligned} \Gamma_{\rm CTP} &= \frac{\lambda^2}{4\pi^3}\int dt \int dt' q^{\rm a}(t)q^{\rm b}(t')\int_0^\infty dz(1-f(2z/\lambda))\psi_{\rm ab} \\ &= \frac{\lambda^2}{4\pi^3}\int_0^\infty dz(1-f(2z/\lambda))\left[\int dt\int dt' q^+(t)q^+(t')\psi_{++} - \int dt \int dt' q^+(t)q^-(t')\psi_{+-}\right. \\ &+ \left.\int dt\int dt' q^-(t)q^+(t')\psi_{-+} - \int dt\int dt' q^-(t)q^-(t')\psi_{--}\right], \end{aligned} \tag{92}$$

where the CTP propagators are

$$\psi_{\pm\pm}(t,t') = \pm\frac{d^2}{dt^2}\int\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{\omega^2-z^2\pm i\epsilon},$$
$$\psi_{\pm\mp}(t,t') = \mp\frac{d^2}{dt^2}\int\frac{d\omega}{2\pi}e^{i\omega(t-t')}2\pi i\delta(\omega^2-z^2)\theta(\pm\omega), \qquad (93)$$

and satisfy the identities

$$\psi_{++} = \psi_{+-}\theta(t-t') - \psi_{-+}\theta(t'-t),$$
$$\psi_{--} = \psi_{-+}\theta(t-t') - \psi_{+-}\theta(t'-t). \qquad (94)$$

Introducing the new variables $\Sigma = (q^+ + q^-)/2$ and $\Delta = (q^+ - q^-)/2$, the CTP effective action can be written as

$$\Gamma_{\rm CTP} = \int dt\int dt'\left[\ddot{\Sigma}(t)\Sigma(t')D(t'-t) - i\ddot{\Delta}(t)\Delta(t')N(t-t')\right], \qquad (95)$$

where the dissipation (D) and noise (N) kernels are given by

$$D(t-t') = \frac{2\lambda^2}{\pi^2}\int_0^\infty dz\ \ (1-f(2z/\lambda))\ \ {\rm Re}\psi_{++}(t,t')\ \theta(t'-t),$$
$$N(t-t') = \frac{\lambda^2}{\pi^2}\int_0^\infty dz\ \ (1-f(2z/\lambda))\ \ {\rm Im}\psi_{++}(t,t'). \qquad (96)$$

Performing the integral in the spectral parameter z we find

$$D(r) = \frac{\lambda^2}{2\pi}\left[(\frac{r\lambda}{2}+1){\rm ChI}(\frac{r\lambda}{2}) + \cos(\frac{r\lambda}{2}) - \sinh(\frac{r\lambda}{2}) - (\frac{r\lambda}{2}+1){\rm ShI}(\frac{r\lambda}{2})\right], \qquad (97)$$

and

$$N(r) = \frac{-\lambda^2}{48\pi^2}\left[-\pi^2 - 24\gamma\ln(\frac{2}{\lambda}) + 12\ln^2(2) - 12\ln^2(\lambda) - 24\ln(\frac{2}{\lambda})\left(\ln(r\lambda) - {\rm ChI}(\frac{r\lambda}{2})\right)\right.$$
$$\left. + \ 24(\ln(\frac{2}{\lambda})-1)\left(\cosh(\frac{r\lambda}{2}) - \frac{r\lambda}{2}{\rm ShI}(\frac{r\lambda}{2})\right)\right] \qquad (98)$$

where $r = |t-t'|$, and ChI, ShI are the hyperbolic CosIntegral and SinIntegral, respectively.

From Eq.(95) we can also see that the dissipative force is given by

$$F_{\rm diss} = \int_{-\infty}^t dt'\ddot{q}(t')D(t-t'). \qquad (99)$$

Taking into account that dissipation kernel has the form $D(t-t') = \lambda^2 g(\lambda(t-t'))$, we can rewrite the force as

$$F_{\rm diss} = \lambda\int_0^\infty dx\, D(x)\ddot{q}(t-\frac{x}{\lambda}) \simeq \lambda\int_0^\infty dx\, D(x)\left[\ddot{q}(t) - \frac{x}{\lambda}q^{(3)}(t) + ...\right]. \qquad (100)$$

A numerical evaluation of the remaining integrals gives the correct perfect conductor limit: the term proportional to λ renormalizes the mass of the mirror, and the λ-independent term gives a dissipation force proportional to the third derivative of the mirror's position.

In order to derive the semiclassical Langevin equation that describes the motion of the mirror, one can regard the imaginary part of $\Gamma_{\rm CTP}$ as coming from a noise source $\eta(t)$ with Gaussian functional probability distribution given by

$$P[\eta(t)] = N_\eta \exp\left\{-\frac{1}{2}\int dt \int dt' \left[\eta(t)\ddot{N}^{-1}(t-t')\eta(t')\right]\right\}, \tag{101}$$

where N_η is a normalization factor. Indeed, we can write the imaginary part of the CTP-effective action as a functional integral over the Gaussian field $\eta(t)$

$$\int \mathcal{D}\eta(t)P[\eta]\; e^{-i\Delta(t)\eta(t)} = e^{-i\int dt\int dt'\Delta(t)\ddot{N}(t-t')\Delta(t')}. \tag{102}$$

Therefore, the CTPEA can be rewritten as an average over the noise of

$$\Gamma^{(\eta)}_{\rm CTP} = \int dt \int dt' \left[\ddot{\Sigma}(t)\Sigma(t')D(t'-t)\right] - \int dt\Delta(t)\eta(t). \tag{103}$$

Thus, the associated Langevin equation comes from the variation $\frac{\delta\Gamma^{(\eta)}_{\rm CPT}}{\delta q_+}|_{q_+=q_-} = 0$, obtaining

$$M\left(\ddot{q}(t) + \Omega^2 q^2(t)\right) + 2\int dt'\ddot{D}(t-t')q(t') = \eta(t), \tag{104}$$

where M is the mass of the mirror and the two-point correlation function of the noise is given by

$$\langle\eta(t)\eta(t')\rangle = \ddot{N}(t-t'). \tag{105}$$

The Langevin equation describes the motion of the system (the mirror) taking into account the main effects of the environment (the quantum field): a dissipative force and a stochastic noise.

A. Mirror's decoherence

In the quantum open system approach that we have adopted here, the imaginary part of the CTP-effective action (noise term) is directly associated with the decoherence process of the mirror. In fact, one can establish a direct link between the total number of created particles and the decoherence functional for a given classical (macroscopic) trajectory of the mirror.

Decoherence means physically that the different coarse-graining histories making up the full quantum evolution acquire individual reality, and may therefore be assigned definite probabilities in the classical sense. For our particular application, we wish to consider as a single coarse-grained history all those fine-grained ones where the trajectory $q(t)$ remains close to a prescribed classical configuration $q_{\rm cl}$.

In principle, we can examine adjacent general classical solutions for their consistency but, in practice, it is simplest to restrict ourselves to particular solutions $q^{\pm}_{\rm cl}$, according to the nature of the decoherence that we are studying. Therefore, we evaluate the decoherence functional [20] for classical trajectories such that the amplitude of one trajectory is $q^- = q^+ - 2\delta$, where δ is a small (constant) amplitude difference. Thus, neglecting the dissipation we can write $\Delta_{\rm cl}(t) = \delta\cos(\Omega t)$, and the decoherence functional is formally given by

$$|\mathcal{D}(q^+_{\rm cl}, q^-_{\rm cl})| = e^{-{\rm Im}\Gamma_{\rm CTP}} = e^{-\delta^2\Omega^2\int dt\int dt'\;\cos(\Omega t)N(t-t')\cos(\Omega t')}. \tag{106}$$

Making the integration in the particular case $m = 0$ and $\lambda = \infty$, one can show that the decoherence time scales as

$$t_D \sim \frac{1}{\delta^2 \omega^3}. \tag{107}$$

This result is valid as long as the decoherence time is much shorter than the dissipative time t_{diss}, which can be easily estimated from Eq.(104) as $t_{\rm diss} \sim M/\Omega^2$. The condition $t_D \ll t_{\rm diss}$ is satisfied as long as $\delta \gg \sqrt{M\Omega}$, i.e the minimum uncertainty in the position of the mirror. An alternative estimation based on the Fokker-Planck equation for the Wigner function of the mirror gives the same order of magnitude for the decoherence time t_D [21].

VI. FINAL REMARKS

In this paper we have extended, in several directions, the functional approach to the dynamical Casimir effect introduced some years ago by Golestanian and Kardar [9] to consider different situations. The main point in this approach, namely, the introduction of auxiliary fields in the functional integral to impose the boundary conditions on the quantum fields is retained, altough now they have an extra piece in the action, to cope with the dispersive nature of the mirror.

After integration of the original quantum fields, the problem is again reduced to the computation of a path integral over the auxiliary fields. This is a kind of (non local) dimensionally reduced theory, since the auxiliary field live on the boundary.

Firstly, we considered non-perfectly conducting mirrors, by introducing a δ-like potentials for the quantum fields. As shown in Ref.[14] for the scalar field, these potentials serve as toy models to describe the interaction of the electromagnetic field with a thin plasma sheet, and give rise to reflection and transmission coefficients with a particular frequency dependence. We believe that this generalization will be useful as a first step towards solving more realistic situations. Indeed, our formalism can be extended to include arbitrary reflection and transmission coefficients by considering non local extensions of the singular potentials considered here. We will describe this results in a forthcoming publication.

We also considered a scalar field coupled to a relativistic mirror, and also calculated the effective action for a mirror interacting with a Dirac spinor, understanding here by mirror an object that reflects the fermionic current.

Finally, we also extended the formulation to calculate the CTP effective action. As an important by-product, we applied this effective action to compute the semiclassical Langevin equation that describes the dynamics of the mirror interacting with the vacuum fluctuations of the quantum fields, and with the motion induced radiation produced by its accelerated motion.

Appendix A: Massive case

We outline here the calculation of the Euclidean effective action in the massive case, for perfectly conducting mirrors in $1+1$ dimensions.

In the definition of $\tilde{F}(\omega)$,

$$\tilde{F}(\omega) = \frac{1}{2}\int_0^\infty \frac{d\nu}{2\pi}\sqrt{\nu^2+m^2}\sqrt{(\nu+\omega)^2+m^2}\,, \tag{108}$$

we insert the representation

$$(\nu^2+m^2)^\epsilon = \frac{1}{\Gamma(-\epsilon)}\int_0^\infty \frac{d\beta}{\beta}\beta^{-\epsilon}e^{-\beta(\nu^2+m^2)}, \tag{109}$$

and one with a shifted argument for the second factor, to obtain the function

$$\tilde{F}_\epsilon(\omega) = \frac{1}{2[\Gamma(-\epsilon)]^2} \int_0^\infty \frac{d\alpha_1}{\alpha_1} \int_0^\infty \frac{d\alpha_2}{\alpha_2} \alpha_1^{-\epsilon} \alpha_2^{-\epsilon} \int_0^\infty \frac{d\nu}{2\pi} \exp\left[-\alpha_1(\nu^2 + m^2) - \alpha_2((\nu + w)^2 + m^2)\right] . \tag{110}$$

The role of this representation is, to make it possible to integrate over the frequency; in a way, it allows for the introduction of Feynman-like parameters in the case of propagators which have a non-standard form. Indeed, introducing the identity

$$1 = \int_0^\infty d\rho\, \delta(\rho - \alpha_1 - \alpha_2)\,, \tag{111}$$

and after a rescaling of the α's plus some straightforward calculations we see that

$$\tilde{F}_\epsilon(\omega) = \frac{\Gamma(-2\epsilon - 1/2)}{4\sqrt{\pi}[\Gamma(-\epsilon)]^2} \int_0^1 d\alpha[\alpha(1-\alpha)]^{-1-\epsilon}[\alpha(1-\alpha)\omega^2 + m^2]^{2\epsilon+1/2} \tag{112}$$

The final result (31) is obtained by subtracting $\tilde{F}_\epsilon(0)$ and then taking the limit $\epsilon = 1/2$. Note that the previously used representation is not used as a regularization, but as a device to do the integral. Indeed, one could have worked with $\epsilon = 1/2$ throughout in the subtracted integral; no analytic extension to complex values of ϵ would be required.

The result for the massless case in $d + 1$ dimensions, (78), can be derived using a similar procedure, but now introducing a dimensional regularization for the momentum integral. Now a regularization *is* required, since the integral over the parallel momenta is divergent when $\epsilon = 1/2$.

Appendix B: Spectral decomposition

In this Appendix we derive the spectral decomposition which allows us to write the form factor $F(\tau - \tau')$ in terms of the Euclidean propagator. Using the identities

$$|\omega| = \frac{2\omega^2}{\pi} \int_0^{+\infty} dz \frac{1}{\omega^2 + z^2}, \tag{113}$$

$$\left(1 + \frac{2}{\lambda}|\omega|\right) \ln\left(1 + \frac{2}{\lambda}|\omega|\right) = \omega^2 \frac{8}{\lambda^2\pi} \int_0^{+\infty} \frac{f(z)}{z^2 + \frac{4\omega^2}{\lambda^2}} \ , \tag{114}$$

where

$$f(z) = \frac{\arctan z}{z} + \frac{1}{2}\ln(1 + z^2) \ , \tag{115}$$

we can write

$$F(\tau - \tau') = -\frac{d^2}{d\tau^2} G(\tau - \tau') \tag{116}$$

with

$$\begin{aligned} G(\tau) &= \int \frac{d\omega}{2\pi} e^{i\omega\tau} \tilde{G}(\omega) \\ \tilde{G}(\omega) &= \frac{\lambda^2}{4\pi^3} \int_0^{+\infty} dz \left\{ \frac{1}{\omega^2 + z^2} \left[1 - f\left(\frac{2z}{\lambda}\right) \right] \right\} . \end{aligned} \tag{117}$$

From these equations the form factor $G(\tau)$ can readily be written in terms of the $0+1$ Euclidean propagator $G_{\rm E}(\tau, z^2)$ with mass z^2

$$G(\tau) = \frac{\lambda^2}{4\pi^3}\int_0^{+\infty} dz \left[1 - f\left(\frac{2z}{\lambda}\right)\right] G_{\rm E}(\tau, z^2) \tag{118}$$

where

$$G_{\rm E}(\tau, z^2) = \int \frac{d\omega}{2\pi}\frac{e^{i\omega\tau}}{\omega^2 + z^2}. \tag{119}$$

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